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PHYSICS AND MATHS

Application of the Fourier Transform to the Solution of a One-Dimensional Non-Stationary Heat Conduction Problem

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Abstract. This article examines the application of the Fourier transform to the analytical solution of a one-dimensional unsteady heat conduction problem in an unbounded homogeneous rod. The original parabolic heat conduction equation is transformed with respect to the spatial coordinate into an ordinary differential equation with respect to time, after which an inverse Fourier transform is performed, yielding an integral representation of the solution in terms of the heat conduction kernel. An explicit analytical formula is derived for a localised Gaussian initial temperature distribution. Based on this, a parametric numerical experiment is carried out, allowing one to track the change in the maximum temperature and the expansion of the heated region over time, as well as to assess the influence of the thermal conductivity coefficient on the rate at which the temperature field equilibrates. The relationships obtained can be used both as a reference analytical solution for verifying numerical methods and as a basic model for further investigation of more complex heat conduction problems involving inhomogeneous properties and boundary conditions.

Keywords: *heat conduction, Fourier transform, heat conduction equation, transient process, thermal conductivity, analytical solution, Gaussian distribution.*

Introduction

Mathematical modelling of heat transfer processes is one of the fundamental problems in mathematical physics, thermophysics and engineering analysis. The classical heat conduction equation describes the diffusive propagation of a temperature disturbance and serves as the basis for studying the heating and cooling of solids, the heat treatment of materials, the operation of power plant components and a range of technological processes. For linear problems with constant thermal coefficients, analytical methods are of particular importance, as they enable the structure of the solution to be identified, the influence of parameters to be assessed, and reference relationships to be obtained for the verification of computational algorithms [1-3].

The Fourier transform is a natural tool for problems in which the spatial domain is unbounded or admits a spectral representation. Its application replaces differentiation with respect to the spatial variable by multiplication by a power function of the spectral variable, thereby reducing the partial differential equation to a simpler time-dependent differential equation. Recent research confirms the relevance of spectral and integral approaches to heat conduction problems. In particular, localised Fourier expansions are applied to two-dimensional non-stationary problems involving complex geometries [4], integral transforms are used for non-linear heat conduction with temperature-dependent properties [5], whilst combinations of Fourier and Laplace transforms enable analytical solutions to be obtained for functional-gradient materials [6]. In 2025, a solution was also proposed for a one-dimensional transient model with a time-varying boundary temperature, based on the properties of the Fourier transform [7].

The Kazakhstani mathematical school is also interested in heat conduction problems with discontinuous coefficients and coupling conditions. In the works of U. K. Koylyshov, M. A. Sadybekov and K. A. Beisenbaeva, the Fourier method is applied to the heat conduction equation for a composite rod with different thermophysical characteristics [8]. This demonstrates that the classical spectral approach remains relevant not only in educational models but also in the study of significantly more complex initial-boundary value problems.

The aim of this work is to derive and analyse an analytical solution to a one-dimensional non-stationary heat conduction problem using the Fourier transform method for a localised initial temperature perturbation.

To achieve this aim, the following tasks are addressed: a mathematical model is formulated; a Fourier transform is performed with respect to the spatial coordinate; a general integral solution is derived; a Gaussian initial temperature distribution is considered; a numerical experiment is carried out and the influence of the thermal conductivity coefficient on the evolution of the temperature field is analysed.

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Problem statement and mathematical model

Consider a homogeneous, unbounded rod in which heat propagates only along the longitudinal coordinate x. The thermophysical properties of the material are assumed to be constant, there are no internal heat sources, and heat exchange with the environment via the lateral surface is neglected. Then, from the law of conservation of energy and Fourier's law for heat flux, we obtain the one-dimensional heat conduction equation

$$\partial T(x,t)/\partial t = \alpha\ \partial^2 T(x,t)/\partial x^2, \quad -\infty < x < \infty, \quad t > 0, \quad (1)$$

where $T(x,t)$ is the temperature, °C; x is the spatial coordinate, m; t is time, s; $\alpha = \lambda/(\rho c)$ is the thermal conductivity coefficient, m^2/s; λ is the thermal conductivity; ρ is the density; c is the specific heat capacity.

We shall define the initial condition in general form as

$$T(x,0) = T_e + f(x), \quad (2)$$

where T_e is the background (equilibrium) temperature, and $f(x)$ is a localised initial temperature perturbation. To ensure the validity of the Fourier transform, we shall assume that $f(x)$ is absolutely integrable or decays sufficiently rapidly as $|x| \to \infty$. Additionally, we assume that $T(x,t) \to T_e$ as $|x| \to \infty$.

Let us introduce the excess temperature $u(x,t) = T(x,t) - T_e$. The problem then takes the form

$$\partial u/\partial t = \alpha\ \partial^2 u/\partial x^2, \quad u(x,0) = f(x), \quad u(x,t) \to 0 \text{ as } |x| \to \infty. \quad (3)$$

Application of the Fourier transform

Let us define the direct Fourier transform with respect to the spatial variable x in the form

$$\hat{u}(k,t) = \int_{-\infty}^{\infty} u(x,t) e^{(-ikx)}\, dx, \quad (4)$$

and the inverse transform by the expression

$$u(x,t) = (1/2\pi) \int_{-\infty}^{\infty} \hat{u}(k,t) e^{(ikx)}\, dk. \quad (5)$$

If the function u and its derivatives decay sufficiently rapidly at infinity, the property of the second derivative transformation holds

$$F\{\partial^2 u/\partial x^2\} = -k^2\hat{u}(k,t). \qquad (6)$$

Applying the Fourier transform to both sides of equation (3), we obtain an ordinary differential equation in t for each fixed value of the spectral variable k:

$$\partial\hat{u}(k,t)/\partial t = -\alpha k^2\hat{u}(k,t). \qquad (7)$$

The initial condition in the spectral domain is given by $\hat{u}(k,0) = \hat{f}(k)$. The solution to equation (7) is found by separation of variables:

$$\hat{u}(k,t) = \hat{f}(k)e^{(-\alpha k^2 t)}. \qquad (8)$$

After applying the inverse Fourier transform, we obtain

$$u(x,t) = (1/2\pi) \int_{-\infty}^{\infty} \hat{f}(k)e^{(-\alpha k^2 t)}e^{(ikx)}\, dk. \qquad (9)$$

The factor $e^{(-\alpha k^2 t)}$ illustrates the physical significance of the diffusion process in the spectral domain: high-frequency spatial components, corresponding to large $|k|$, decay more rapidly than low-frequency ones. This is precisely why the temperature field smooths out over time, and sharp spatial inhomogeneities disappear.

Using the convolution theorem, expression (9) can be rewritten in terms of the fundamental solution to the heat conduction equation — the Gaussian kernel

$$G(x,t) = 1/\sqrt{(4\pi\alpha t)} \cdot \exp[-x^2/(4\alpha t)]. \qquad (10)$$

Then the general solution to the Cauchy problem takes the form

$$u(x,t) = \int_{-\infty}^{\infty} G(x-\xi,t)\, f(\xi)\, d\xi. \qquad (11)$$

Equation (11) demonstrates that the temperature at point x at time t is a weighted sum of all values of the initial temperature field. The weight is determined by a Gaussian kernel, the width of which increases proportionally to $\sqrt{(\alpha t)}$.

Partial solution for a Gaussian initial distribution

To obtain an explicit formula, let us define the initial temperature perturbation as

$$f(x) = \Delta T_0 \exp(-x^2/b^2), \tag{12}$$

where ΔT_0 is the maximum temperature excess above the background level at the initial moment, and $b > 0$ characterises the spatial width of the heated region. The Fourier transform of function (12) is known in analytical form:

$$\hat{f}(k) = \Delta T_0\, b\sqrt{\pi} \cdot \exp(-b^2k^2/4). \tag{13}$$

Substituting (13) into (9), we combine the exponential terms and apply the standard Gaussian integral. As a result, we obtain

$$u(x,t) = \Delta T_0 \cdot b/\sqrt{(b^2+4\alpha t)} \cdot \exp[-x^2/(b^2+4\alpha t)]. \tag{14}$$

Consequently, the physical temperature is given by the expression

$$T(x,t) = T_e + \Delta T_0 \cdot b/\sqrt{(b^2+4\alpha t)} \cdot \exp[-x^2/(b^2+4\alpha t)]. \tag{15}$$

Equation (15) reveals two interrelated effects. Firstly, the effective width of the temperature spot increases as $\sqrt{(b^2 + 4\alpha t)}$. Secondly, the maximum excess temperature at the centre, $x = 0$, decreases according to the law

$$T(0,t) - T_e = \Delta T_0 \cdot b/\sqrt{(b^2+4\alpha t)}. \tag{16}$$

In this case, the integral $\int_{-\infty}^{\infty} u(x,t)\,dx$ is conserved over time, which corresponds to the conservation of total excess thermal energy in the idealised model of an unbounded rod with no heat exchange with the external environment.

Numerical example and results

To illustrate the analytical solution, let us take $T_e = 20$ °C, $\Delta T_0 = 80$ °C, $b = 0.02$ m and $\alpha = 1.0 \times 10^{-5}$ m²/s. These values are used as a model set of parameters and are not specific to any particular material. The calculation is performed directly using equation (15); therefore, the numerical error is associated only with the construction of discrete graphs, and not with the solution of the differential equation.

The calculated values are presented in Table 1.

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Table 1

Change in temperature at the centre and characteristic width of the temperature profile

t, s	T(0,t), °C	$\sqrt{(b^2+4\alpha t)}$, m
0	100.00	0.0200
10	76.57	0.0283
30	60.00	0.0400
60	50.24	0.0529
100	44.12	0.0663

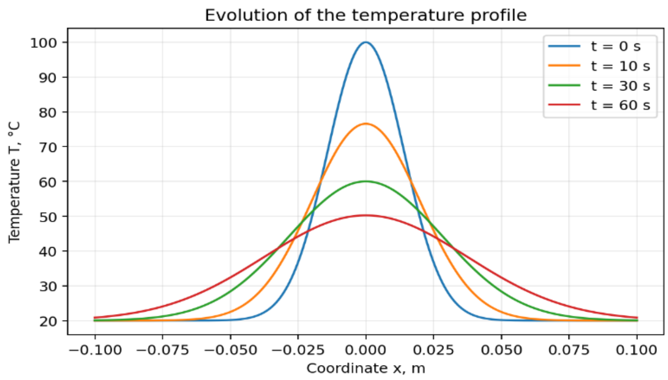

Figure 1
Temperature distribution along the rod at various points in time

Figure 1 shows that the initial narrow temperature maximum gradually decreases whilst simultaneously broadening. At t = 10 s, the central temperature decreases to approximately 76.57 °C; at t = 30 s, to 60.00 ° C; and at t = 60 s, to 50.24 °C. Despite the decrease in the peak, the region of noticeable temperature deviation from the background level becomes wider, which directly reflects the diffusive nature of heat transfer.

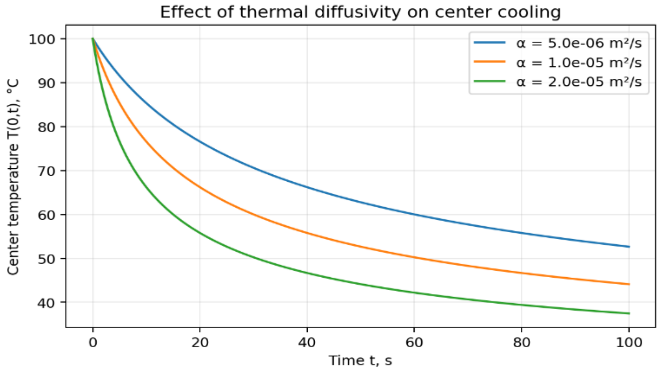

Figure 2
Effect of the thermal conductivity coefficient on the temperature at the centre of the heated zone

Figure 2 compares three values of thermal conductivity: $0.5 \cdot 10^{-5}$, $1.0 \cdot 10^{-5}$ and $2.0 \cdot 10^{-5}$ m²/s. The higher the value of α, the faster the temperature at the centre decreases, as the thermal disturbance is redistributed more intensively throughout the space. Thus, for identical initial parameters, doubling α leads to a more rapid expansion of the temperature profile and, consequently, to a more rapid decrease in its amplitude.

Analysis and discussion of the results

The solution obtained allows the characteristic scales of the process to be determined without the use of complex computational procedures. If we introduce the dimensionless time $Fo = \alpha t/b^2$, analogous to the Fourier number for the chosen scale b, then equation (16) takes the form $(T(0,t) - T_e)/\Delta T_0 = 1/\sqrt{(1 + 4Fo)}$. Consequently, the rate of relative decrease in the central temperature is determined not by the individual parameters α, t and b, but by their dimensionless combination Fo. This makes the solution suitable for generalising the results to different materials and sizes of the initial heated region.

The spectral representation also provides a clear interpretation of the smoothing mechanism. According to equation (8), each harmonic of the initial field decays with a factor of $e^{(-\alpha k^2 t)}$. Therefore, short-wavelength components disappear more rapidly than large-scale components. In physical space, this manifests as the disappearance of sharp temperature gradients. This relationship between spectral attenuation and diffusion smoothing is one of the main advantages of the Fourier method in the qualitative analysis of parabolic equations.

From a practical point of view, equation (15) can be used as a test solution when verifying programmes that implement finite difference, finite element and spectral methods. If the numerical scheme is correct and the mesh sizes are sufficiently small, its result should converge to the analytical curve. Recent work on localised Fourier methods and integral transforms also utilises analytical or reference solutions to assess the accuracy of new computational approaches [4,5].

The model's limitations stem from the assumptions made. The constant value of α means that the temperature dependence of thermophysical properties is not taken into account; the infinite domain excludes the influence of boundaries; and there are no internal sources or convective heat transfer.

For real bodies, these factors may be significant. In particular, the problem involving temperature-dependent thermal conductivity becomes non-linear and requires generalised integral transforms or numerical methods [5], whilst composite media necessitate the consideration of coupling conditions and discontinuous coefficients [8]. Therefore, the solution considered here should be regarded as a basic model, from which it is natural to proceed to more complex formulations within the framework of the doctoral research.

The scientific and practical significance of this work lies in the step-by-step derivation of an exact solution for localised initial heating and in the parametric analysis of its evolution via the thermal conductivity coefficient and the dimensionless time Fo. This approach links the analytical part of the thesis with subsequent numerical modelling: equation (15) can serve as a test solution, and the same methodology can then be extended to finite domains, inhomogeneous coefficients and mixed boundary conditions.

Conclusion

This work examines the application of the Fourier transform to the one-dimensional unsteady heat conduction equation in an unbounded homogeneous medium. Transformation with respect to the spatial coordinate made it possible to reduce the original partial differential equation to a family of ordinary differential equations in the spectral domain and to obtain a general solution via the Gaussian heat conduction kernel.

For a Gaussian initial temperature distribution, an explicit analytical formula has been derived describing the simultaneous decrease in amplitude and broadening of the temperature profile. A numerical example demonstrated that an increase in the thermal conductivity coefficient accelerates the equalisation of the temperature field. The introduction of the dimensionless time $Fo = \alpha t/b^2$ allows the results to be presented in a universal form and enables the comparison of various physical parameters on a single scale.

The solution obtained is suitable for use as the core section of a doctoral thesis and as a reference model for subsequent research into heat conduction problems over a finite interval, in composite media, with varying thermophysical coefficients and under more complex boundary conditions. A promising avenue for further work is to compare the analytical solution with finite-difference results and to investigate the accuracy of the numerical approximation for various spatial and temporal steps.

References:

[1] Carslaw H.S., Jaeger J.C. Conduction of Heat in Solids. 2nd ed. Oxford: Clarendon Press, 1959. 510 p.

[2] Özisik M.N. Heat Conduction. 2nd ed. New York: John Wiley & Sons, 1993. 692 pp.

[3] Evans L.C. Partial Differential Equations. 2nd ed. Providence, RI: American Mathematical Society, 2010. 749 pp.

[4] Li X., Zhao S., Qu W. Localised Fourier collocation method for 2D transient heat conduction problems // Applied Mathematics Letters. 2025. Vol. 160. Article 109331. DOI: 10.1016/j.aml.2024.109331.

[5] Yu J., An C., He Y., Su J. Transient nonlinear heat conduction in a slab with temperature-dependent thermal conductivity: Integral transform and lumped model solutions // International Communications in Heat and Mass Transfer. 2024. Vol. 158. Article 107886. DOI: 10.1016/j.icheatmasstransfer.2024.107886.

[6] Delouei A.A., Emamian A., Ghorbani S., He F. A general unsteady Fourier solution for orthotropic heat transfer in 2D functionally graded cylinders // Mathematical Methods in the Applied Sciences. 2024. Vol. 47, No. 6. pp. 3942-3959. DOI: 10.1002/mma.9796.

[7] Wu D., Wang Y., Tao Y., Li X., Ren H., Yang Q. A Fast Fourier Transform Solution for a 1D Unsteady Heat Transfer Model Bounded by a Varying Temperature // Archives of Thermodynamics. 2025. Vol. 46, No. 2. pp. 167-172. DOI: 10.24425/ather.2025.154915.

[8] Koilyshov U.K., Sadybekov M.A., Beisenbayeva K.A. Solution of an initial-boundary value problem for the heat equation with a discontinuous coefficient and a general conjugation condition // Filomat. 2025. Vol. 39, No. 23. pp. 7997-8005. DOI: 10.2298/FIL2523997K.

[9] Crank J. The Mathematics of Diffusion. 2nd ed. Oxford: Clarendon Press, 1975. 414 pp.

[10] Beck J.V., Cole K.D., Haji-Sheikh A., Litkouhi B. Heat Conduction Using Green's Functions. 2nd ed. Boca Raton: CRC Press, 2011. 694 pp.